ADOMIAN DECOMPOSITION METHOD WITH NEUMANN BOUNDARY CONDITIONS FOR SOLUTION OF NONLINEAR BOUNDARY VALUE PROBLEM

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Abstract: *The Adomian decomposition method (ADM) is a creative and effective method for exact solution of functional equations of various kinds. Adomian decomposition method solves wide class of linear and non-linear, ordinary or partial differential equations. This paper presents the Adomian decomposition method for the solution of nonlinear boundary value problem using Neumann boundary conditions. In this approach, the solution is found in the form of a convergent power series with easily computed components. To show the efficiency of the method, numerical results and graphical representation of results are presented and compared with exact solution.*

Keywords: Adomian decomposition method, Neumann boundary conditions, Nonlinear boundary value problem, Bratu problem, Burger problem.

1. INTRODUCTION

The Adomian Decomposition Method [1-3] has been utilized for the solution of linear and nonlinear ordinary and partial differential equations. Some authors known as Kaya [4], El-Sayed, Biazar Hashim, and Lesnic [5-7] examined some different scientific models analytically and numerically. Another two writers named Sweilam and Khader [8-10] used the ADM for the analysis of nonlinear atmosphere of multi-walled carbon Nano tubes. ADM moreover gives a convergent sequence of estimations for some terms of high accuracy. Cherruault [11, 12] discussed the convergence of the ADM. This method was proved very valuable for solving boundary value problems by many other writers. There was a reason of its value, which was shortest use of restraining supposition, linearization or Green functions. Rach and Adomian [13, 14] exemplified the way to solve nonlinear BVPs in only some measurement with the help of decomposition technique and considered many ordinary and partial differential equations through the two conditions known as Dirichlet and Neumann boundary [15].

2. MATERIAL AND METHODS

Let the two-point boundary value problem be

$$u''(x) + p(x)u'(x) + q(x)f\big(u(x)\big) = r(x) \qquad (1)$$

$x \in [a, b]$,

Along with the Neumann boundary conditions

$$u'(a) = \alpha\,, u'(b) = \beta \qquad (2)$$

Suppose the inverse operator

$$L_{xx}^{-1}[.] = \iint [.]dxdx$$

is relate to Eq. (1) resultant in

$$u(x) = \varphi_x + L_{xx}^{-1}(r(x) - p(x)u'(x) - q(x)f\big(u(x)\big)(3)$$

where $\varphi_x = c_1 + c_2 x, c_1$ *and* c_2 are constants, the linear and nonlinear terms *u* and *f (u(x))* as follows:

$$\varphi_x = \sum_{n=0}^{\infty} \varphi_{x,n} = \sum_{n=0}^{\infty}(c_{1,n} + c_{2,n}x),$$
$$u(x) = \sum_{n=0}^{\infty} u_n(x),$$
$$f\big(u(x)\big) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \cdots, u_n)\} \qquad (4)$$

where $A_n$the Adomian polynomials that are computed as follows

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n} f(\sum_{i=0}^{\infty} \lambda^i u_i)]_{\lambda=0}\,,\ n \geq 0$$

The decomposition of the initial expression is required substitute (4) into (3),

$$u_0(x) = \varphi_{x,0} + L_{xx}^{-1}r(x)$$
$$u_{n+1}(x) = \varphi_{x,n} - L_{xx}^{-1}(p(x)u_n'(x) + q(x)A_n), \quad n \geq 0$$

The Eq. (1) is given by

$$\varphi_m(x) = \sum_{n=0}^{\infty} u_n(x)\ for\ \ m > 0$$

Thus $\varphi_1 = u_0, \varphi_2 = \varphi_1 + u_1, \varphi_3 = \varphi_2 + u_2$, etc.

$\varphi_1 = u_0 = c_{1,0} + c_{2,0}x + L_{xx}^{-1}r(x)$.

Using (2) results in

$$c_{2,0} + [L_{xx}^{-1}r(x)]'_{x=a} = \alpha$$
$$c_{2,0} + [L_{xx}^{-1}r(x)]'_{x=b} = \beta$$

If u′ (a) = α and u′ (b) = β are ordinary differential equation of second order through Neumann boundary conditions

$$L_{xx}^{-1}u''(x) = u(x) - (x - \Omega)u'(a) - \frac{\Omega}{2}u'(b) - \frac{1}{\Omega}\int_0^{\Omega} u(x)dx\,,\ \ a \leq x \leq b$$

where $L_{xx}^{-1}[.]$ is obvious by means of

$$L_{xx}^{-1}[.] = \int_{\Omega}^{x} dx' \int_{a}^{x'} [.]dx'' + \frac{1}{\Omega}\int_0^{\Omega} dx'(x' \int_{b}^{x'} [.]dx'')(5)$$

Where Ω is random

$$L_{xx}^{-1}[.] = \int_{\Omega}^{x} dx' \int_{a}^{x'} [.]dx'' + \frac{1}{\Omega}\int_0^{\Omega} dx'(x' \int_{b}^{x'} [.]dx'')$$

Then

$$L_{xx}^{-1}[u''(x)] = \int_{\Omega}^{x} dx' \int_{a}^{x'} [u''(x)]dx'' + \frac{1}{\Omega}\int_0^{\Omega} dx'(x' \int_{b}^{x'} [u''(x)]dx'')$$
$$= \int_{\Omega}^{x}[u'(x') - u'(a)]dx' + \frac{1}{\Omega}\int_0^{\Omega} x'[u'(x') - u'(b)]dx'$$
$$= u(x) - u(\Omega) - (x - \Omega)u'(a) + \frac{1}{\Omega}\int_0^{\Omega} x'u'(x')dx' - \frac{1}{\Omega}\left[\frac{\Omega^2}{2}u'(b)\right]$$
$$= u(x) - u(\Omega) - (x - \Omega)u'(a) + \frac{1}{\Omega}\int_0^{\Omega} x'u'(x')dx' - \frac{\Omega}{2}u'(b)$$
$$= u(x) - u(\Omega) - (x - \Omega)u'(a) + \frac{1}{\Omega}[\Omega u(\Omega) - \int_0^{\Omega} u(x')dx'] - \frac{\Omega}{2}u'(b)$$
$$= u(x) - u(\Omega) - (x - \Omega)u'(a) + [u(\Omega) - \frac{1}{\Omega}\int_0^{\Omega} u(x')dx'] - \frac{\Omega}{2}u'(b)$$
$$= u(x) - (x - \Omega)u'(a) - \frac{\Omega}{2}u'(b) - \frac{1}{\Omega}\int_0^{\Omega} u(x)dx$$

There is a setup of algorithms used for linear and nonlinear second-order ordinary and partial differential equations by the help of Neumann boundary form.

$$u''(x) = r(x) - p(x)u'(x) - q(x)f\big(u(x)\big)$$

Now, the operator $L_{xx}^{-1}(.)$ by Eq. (5), it will help to obtain

$$u(x) = (x-\Omega)u'(a) + \frac{\Omega}{2}u'(b) + \frac{1}{\Omega}\int_0^{\Omega} u(x)dx + L_{xx}^{-1}\Big(r(x) - p(x)u'(x) - q(x)f\big(u(x)\big)\Big)$$

When u and f (u(x)) are decomposed by Eq. (4)

$$u_0 = (x-\Omega)u'(a) + \frac{\Omega}{2}u'(b) + L_{xx}^{-1}[r(x)]$$

$$u_{n+1} = \frac{1}{\Omega}\int_0^{\Omega} u_n(x)dx - L_{xx}^{-1}[p(x)u_n'(x) + q(x)A_0$$

as well as within the linear form

$$u_0 = (x-\Omega)u'(a) + \frac{\Omega}{2}u'(b) + u_{xx}^{-1}[r(x)]$$

$$u_{n+1} = \frac{1}{\Omega}\int_0^{\Omega} u_n(x)dx - L_{xx}^{-1}[p(x)u_n'(x) + q(x)u_n] \ , \ n \geq 0$$

The estimated analytic explanation is given by

$$\varphi_n(x;\Omega) = \sum_{i=0}^{\infty} u_i(x;\Omega)$$

3. NUMERICAL ILLUSTRATIONS

3.1 Bratu Problem

The following Bratu problem

$$u''(x) - 2e^{u} = 0\,, \qquad 0 \leq x \leq 1$$

$$u'(0) = 0\ ,\ \ u'(1) = 2\tan(1)$$

Application of ADM gives

$$u_0 = (x-\Omega)u'(a) + \frac{\Omega}{2}u'(b) + L_{xx}^{-1}[r(x)]$$

$$u_0 = \Omega\tan(1)$$

$$u_{n+1} = \frac{1}{\Omega}\int_0^{\Omega} u_n(x)dx + \int_{\Omega}^{x} dx' \int_0^{x'} [2A_n(x'')]dx'' + \frac{1}{\Omega}\int_0^{\Omega} dx'(x' \int_1^{x'} [2A_n(x'')]dx'')$$

where n ≥ 0 and A_n be the Adomian polynomials which are

$$A_0 = e^{u_0}$$

$$A_1 = u_1 e^{u_0}$$

$$A_2 = \left(u_2 + \frac{u_1^2}{2}\right)e^{u_0}$$

$$A_3 = \left(u_3 + u_1u_2 + \frac{u_1^3}{6}\right)e^{u_0}$$

The estimated solutions Φ2(x), Φ3(x) and Φ4(x) as $\Omega \to 0$ are given by

$$u_1 = \frac{1}{\Omega}\ \Omega\tan(1)(\Omega) + \int_{\Omega}^{x} dx'[2e^{u_0}]dx'' + 0$$

$$= \Omega\tan(1) + \int_{\Omega}^{x} 2e^{\Omega\tan(1)}dx''$$

$$= \Omega\tan(1) + 2[e^{\Omega\tan(1)}x]_{\Omega}^{x}$$

$$= \Omega\tan(1) + 2e^{\Omega\tan(1)}\left(\frac{x^2}{2}\right)$$

$$u_1 = x^2$$

$$\varphi_2(x) = u_0 + u_1,\ \ \varphi_2(x) = \Omega\tan(1) + x^2$$

$$\varphi_2(x) = x^2$$

$$\varphi_3(x) = x^2 + \frac{x^4}{6}$$

$$\varphi_4(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45}$$

The exact solution is u(x) = −2 log [cos(x)]. The Maclaurin series of this solution is

$$u(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260} + \frac{62x^{10}}{14175} + \cdots.$$

Table-1: Summary of results from Bratu problem

x	U	Exact	Error
0	0	0	0
0.05	.002501041667194	.002501042361638	0.000000000694444
0.1	.010016666802025	.01006711246470	0.000000044444446
0.15	.02258437843090	.022584884733284	0.000000506250193
0.2	.040266701654236	.040269546104816	0.000002844450580
0.25	.063151251710491	.063162102494940	0.000010850784448
0.3	.091350911041714	.091383311852116	0.000032400810402
0.35	.125004200567281	.125085906475610	0.000081705908329
0.4	.164275967461813	.164458038150110	0.000182070688298
0.45	.209358551413617	.209727717240980	0.000369165827363
0.5	.260473641424162	.261168480887446	0.000694839463283
0.55	.317875094898601	.319106610717962	0.000694839463283
0.6	.38185306214400	.383930338838876	0.002077276694875
0.65	.452739843933609	.45610648239232	0.003361804305624
0.7	.530918009062173	.536171515135862	0.005253506073689
0.75	.616831414358957	.624799795798210	0.007968381439254
0.8	.710999902991803	.722781493622688	0.011781590630885
0.85	.814038609637238	.83108196165270	0.017043352014031
0.9	.926682974758286	.950884887171628	0.024201912413343
1	1.18453262786596	1.23125940772020	0.046720312906067

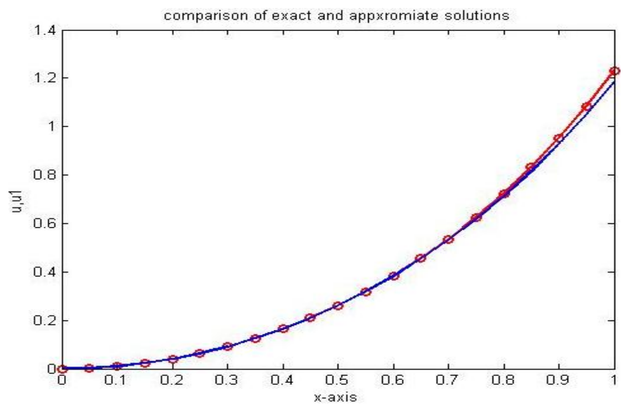

Figure 1: Graph for Bratu problem

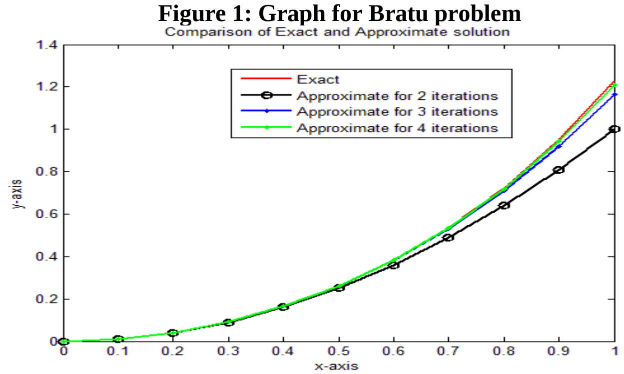

Figure 2: Graph for Bratu problem with iterations

3.2 Burger Problem

Suppose the nonlinear Burger problem

$$u'' + uu' + u = \frac{1}{2}\sin(2x), \qquad 0 \leq x \leq \frac{\pi}{2}$$

$$u'(0) = 1\,,\ \ u'\left(\frac{\pi}{2}\right) = 0$$

Consider the ADM; the solution can be computed as follows

$$u_0 = (x-\Omega)u'(a) + \frac{\Omega}{2}u'(b) + L_{xx}^{-1}[r(x)]$$

$$= (x-\Omega) + L_{xx}^{-1}[\frac{1}{2}\sin(2x)]$$

$$= (x-\Omega) + \int_{\Omega}^{x} dx' \int_0^{x'} [\frac{1}{2}\sin(2x)]dx'' + \frac{1}{\Omega}\int_0^{\Omega} dx'(x' \int_{\frac{\pi}{2}}^{x'} \frac{1}{2}\sin(2x))dx''$$

$$= (x-\Omega+\int_{\Omega}^{x}dx'\left|\frac{1}{2}.\frac{-\cos(2x)}{2}\right|_{0}^{x'}+\frac{1}{\Omega}\int_{0}^{\Omega}dx'\left(x'\left|\frac{1}{2}.\frac{-\cos(2x)}{2}\right|_{\frac{\pi}{2}}^{x'}\right)$$

$$= (x-\Omega)+\int_{\Omega}^{x}dx'(\frac{-1}{4}(\cos(2x')-1))+\frac{1}{\Omega}\int_{0}^{\Omega}dx'(x'(\frac{-1}{4}(\cos(2x')-\cos 2(\frac{\pi}{2}))))$$

$$= (x-\Omega)+\int_{\Omega}^{x}\frac{-1}{4}\cos 2x'dx'+\int_{\Omega}^{x}\frac{1}{4}dx'+\frac{1}{\Omega}\int_{0}^{\Omega}\frac{-1}{4}x'\cos 2x'dx'+\frac{1}{\Omega}\int_{0}^{\Omega}x'dx'$$

$$= (x-\Omega)+\frac{-1}{4}\left|\frac{\sin 2x'}{2}\right|_{\Omega}^{x}+\frac{1}{4}|x'|_{\Omega}^{x}+\frac{1}{\Omega}\left|\frac{-1}{4}x'\frac{\sin 2x'}{2}\right|_{0}^{\Omega}-\int_{0}^{\Omega}\frac{-1}{4}.\frac{\sin 2x'}{2}dx'+\frac{1}{\Omega}.\left|\frac{x'^2}{2}\right|_{0}^{\Omega}$$

$$= (x-\Omega)+\frac{-1}{8}(\sin(2x)-\sin 2\Omega)+\frac{1}{4}(x-\Omega)+\frac{1}{\Omega}(\frac{-1}{8}\Omega\sin 2\Omega)+\frac{1}{8\Omega}.\left|\frac{-\cos 2x}{2}\right|_{0}^{\Omega}+\frac{1}{\Omega}\left(\frac{\Omega^2}{2}\right)$$

$$= (x-\Omega)+\frac{-1}{8}\sin 2x+\frac{1}{8}\sin 2\Omega+\frac{1}{4}x-\frac{1}{4}\Omega-\frac{\sin 2\Omega}{8}-\frac{1}{16\Omega}\cos 2\Omega+\frac{1}{16\Omega}+\frac{\Omega}{2}$$

$$u_0=(x-\Omega)+\frac{-1}{16\Omega}[2\Omega\sin 2x+\cos 2\Omega-4\Omega x-4\Omega^2-1]$$

$$u_1=\frac{1}{\Omega}\int_{0}^{\Omega}u_0(x)dx-L_{xx}^{-1}[u_0+u_0u_0']$$

$$=\frac{1}{\Omega}\int_{0}^{\Omega}xdx-L_{xx}^{-1}[x+(x)(1)]$$

$$=\frac{1}{\Omega}\int_{0}^{\Omega}xdx-L_{xx}^{-1}[2x]$$

$$=\frac{1}{\Omega\left[\frac{x^2}{2}\right]_{0}^{\Omega}}-\int_{\Omega}^{x}dx'\int_{0}^{x'}[2x]dx''+\frac{1}{\Omega}\int_{0}^{\Omega}dx'\left(x'\int_{\frac{\pi}{2}}^{x'}[[2x]dx''])\right]$$

$$=\frac{1}{\Omega}\left[\frac{\Omega^2}{2}\right]-\int_{\Omega}^{x}dx'[(\frac{2x^2}{2})]_{0}^{x'}+\frac{1}{\Omega}\int_{0}^{\Omega}dx'(x'\left[\frac{2x^2}{2}\right]_{\frac{\pi}{2}}^{x'})$$

$$=-[\frac{x^3}{3!}]_{\Omega}^{x}$$

$$\varphi_2(x)=u_0+u_1$$

$$\varphi_2(x)=x-\frac{x^3}{3!}-\frac{3x^5}{40}+\cdots$$

$$\varphi_3(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{109x^7}{5040}+\cdots$$

$$\varphi_4(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}-\frac{703x^9}{120960}+\cdots$$

$$\varphi_5(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}+\frac{888x^{11}}{5702400}+\cdots.$$

Table-2: Summary of results from Burger problem

x	U	Exact	Error(1.0e-07)
0	0	0	0
.05	.049979169270678	.4997916927068	0.000000000069389
.1	.099833416646828	.099833416646828	0.000000000138778
.15	.14938132473599	.149438132473599	0.000000000555112
.2	.198669330795062	.198669330795061	0.000000005273559
.25	.247403959254529	.247403959254523	0.000000059674488
.3	.295520206661384	.295520206661340	0.000000059674488
.35	.342897807455693	.342897807455451	0.000002416400413
.4	.389418342309700	.389418342308651	0.000010496603586
.45	.434965534115064	.434965534111230	0.000038336556152
.5	.479425538616416	.579425538604203	0.000122128973601
.55	.522687228965492	.522687228930659	0.000348330253530
.6	.564642473485714	.564642473395035	0.000906789088262
.65	.605186405954673	.605186405736039	0.002186331116150
.7	.644217687731501	.644217687237691	0.002186331116150
.75	.681638761077608	.681638760023334	0.010542742234776
.8	.717356093042681	.717356090899523	0.021431579844133
.85	.751280409313244	.751280405140293	0.041729514288491
.9	.783326917448438	.783326909627483	0.078209541065632
.95	.813415518956931	.813415504789374	0.141675567943977
1	.841471009700176	.841470984807897	0.248922799039875

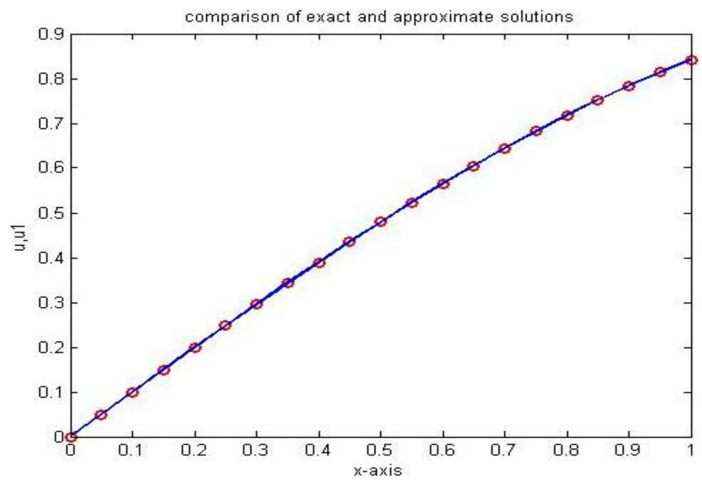

Figure 3: Graph for Burger problem

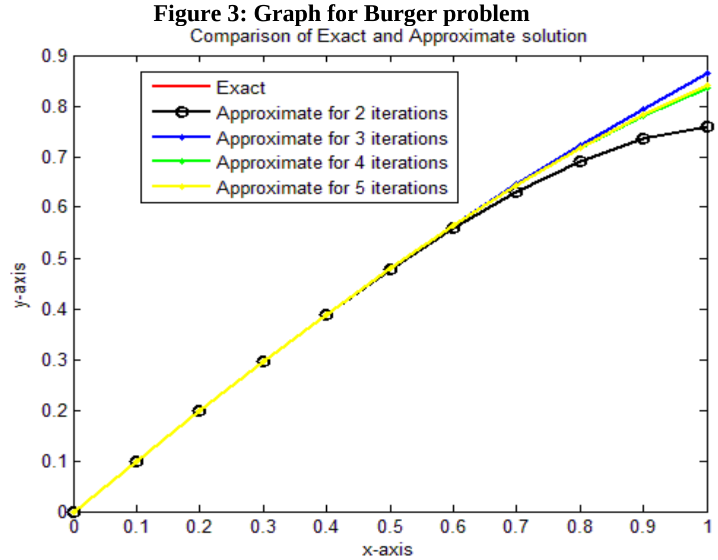

Figure 4: Graph for Burger problem with iterations

3.3 Nonlinear BVP

The following nonlinear BVP

$$u''-(u')^2=0$$

$$u'(0)=-1\ \ ,u'(1)=\frac{-1}{2}$$

We obtain the following

$$A_n=(n+1)\sum_{i=1}^{n}u_{n-i}'u_i'$$

$$u_0=(x-\Omega)u'(a)+\frac{\Omega}{2}u'(b)+L_{xx}^{-1}[r(x)]$$

$$=-x+\Omega-\frac{\Omega}{4}$$

$$u_0=-x+\frac{3\Omega}{4}\ \ ,u_0=-x+\mu_0$$

$$u_0=-x+\mu_0$$

$$u_{n+1}=\int_{\Omega}^{x}(\int_{0}^{x}A_n dx)dx+\frac{1}{\Omega}\int_{0}^{\Omega}(x\int_{1}^{x}A_n dx)dx+\frac{1}{\Omega}\int_{0}^{\Omega}u_n dx$$

The following approximants

$$u_1 = \int_{\Omega}^{x}\int_0^x A_0 dx + \frac{1}{\Omega}\int_0^{\Omega}(x\int_1^x A_0 dx + \frac{1}{\Omega}\int_0^{\Omega} u_0 dx)$$

$$= \int_{\Omega}^{x}\int_0^x (u_0')^2 dx \quad , \int_{\Omega}^{x}\int_0^x dxdx$$

$$u_1 = \frac{x^2}{2} - \frac{\Omega^2}{2}$$

$$u_1 = \frac{x^2}{2} + \mu_1$$

$$\varphi_2(x) = u_0 + u_1$$

$$\varphi_2(x) = -x + \frac{x^2}{2} + \mu_1$$

$$\varphi_2(x) = -x + \frac{x^2}{2} + \mu_1$$

$$\varphi_3(x) = -x + \frac{x^2}{2} - \frac{x^3}{3} + \mu_2$$

$$\varphi_4(x) = -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \mu_3$$

$$\varphi_5(x) = -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \mu_4$$

where μ_i are constants in terms of Ω. As n ⟶ ∞, then u = −log(x+1) +μ, where μ= limit μ_i as n ⟶ ∞

Table 3: Summary of results from above nonlinear BVP

X	u	Exact	Error
0	0	0	0
0.05	-0.048790166666667	-0.048790164169432	0.000000002497235
0.1	-0.095310333333333	-0.095310179804325	0.000000153529008
0.15	-0.139763625000000	-0.139761942375159	0.000001682624841
0.2	-0.182330666666667	-0.182321556793955	0.000009109872712
0.25	-0.223177083333333	-0.223143551314210	0.000033532019124
0.3	-0.262461000000000	-0.262364264467491	0.000096735532509
0.35	-0.300340541666667	-0.300104592450338	0.000235949216329
0.4	-0.336981333333333	-0.336472236621213	0.000509096712120
0.45	-0.372564000000000	-0.371563556432483	0.001000443567517
0.5	-0.407291666666667	-0.405465108108164	0.001826558558502
0.55	-0.441397458333333	-0.438254930931155	0.003142527402178
0.6	-0.475152000000000	-0.470003629245736	0.005148370754264

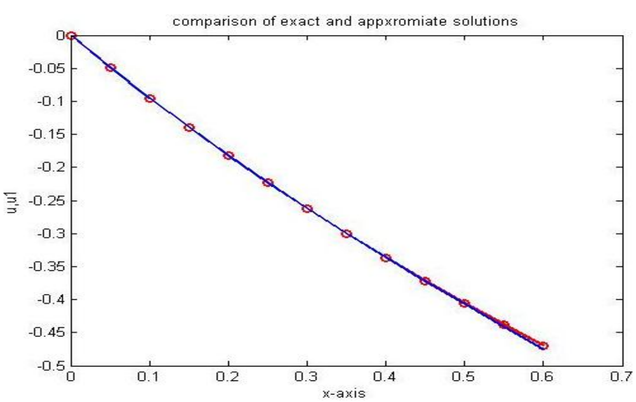

Figure 5: Graph for above nonlinear BVP

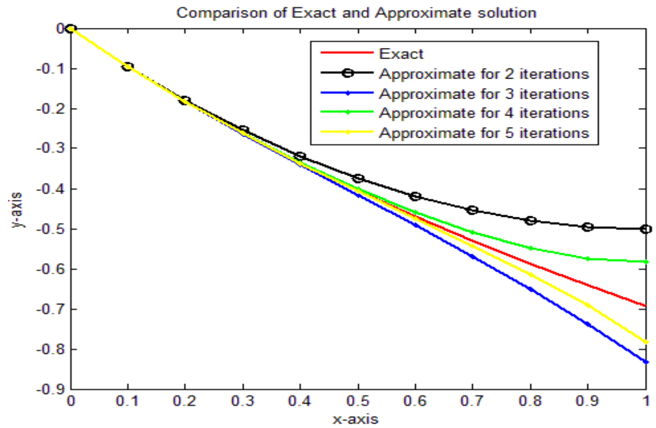

Figure 6: Graph for above nonlinear BVP with iterations

3.4 Nonhomogeneous Wave Equation

Let the nonhomogeneous wave equation be

$$u_{tt} = u_{xx} + 2\pi^2 e^{-\pi t}\sin(\pi x), \qquad 0 \le x \le 1, t \ge 0$$

$$u_x(0,t) = \pi e^{-\pi t}, \quad u_x(1,t) = -\pi e^{-\pi t}$$

We use the recursion method

$$u_0 = (x-\Omega)u'(a) + \frac{\Omega}{2}u'(b) + L_{xx}^{-1}[r(x)]$$

$$= (x-\Omega)\pi e^{-\pi t} + \frac{\Omega}{2}(-\pi e^{-\pi t}) - L_{xx}^{-1}[2\pi^2 e^{-\pi t}\sin(\pi x)]$$

$$= (x-\Omega)\pi e^{-\pi t} + \frac{\Omega}{2}(-\pi e^{-\pi t}) + 2\pi^2 e^{-\pi t}[\int_{\Omega}^{x} dx' \int_0^{x'} \sin(\pi x) dx'' + \frac{1}{\Omega}\int_0^{\Omega} dx'(x'\int_1^{x'} \sin(\pi x)) dx'']$$

$$= (x-\Omega)\pi e^{-\pi t} + \frac{\Omega}{2}(-\pi e^{-\pi t}) + \int_{\Omega}^{x} dx' \left|\frac{-\cos(\pi x)}{\pi}\right|_0^{x'} + \frac{1}{\Omega}\int_0^{\Omega} dx'\left(x'\left|\frac{-\cos(\pi x)}{\pi}\right|_1^{x'}\right)$$

$$= (x-\Omega)\pi e^{-\pi t} + \frac{\Omega}{2}(-\pi e^{-\pi t}) + \int_{\Omega}^{x} dx'\left(\frac{-\cos(\pi x')}{\pi} + \frac{1}{\pi}\right) + \frac{1}{\Omega}\int_0^{\Omega} dx'\left(x'\left(\frac{-\cos(\pi x')}{\pi} - \frac{1}{\pi}\right)\right)$$

$$= (x-\Omega)\pi e^{-\pi t} + \frac{\Omega}{2}(-\pi e^{-\pi t}) + \left|\frac{-\sin(\pi x')}{\pi^2}\right|_{\Omega}^{x} + \frac{1}{\pi}(x-\Omega) + \frac{1}{\Omega}\left|x'.\frac{-\sin(\pi x')}{\pi^2}\right|_0^{\Omega} - \int_0^{\Omega}\frac{-\sin(\pi x')}{\pi^2}dx' - \frac{1}{\pi}\left(\frac{\Omega^2}{2}\right)$$

$$= (x-\Omega)\pi e^{-\pi t} + \frac{\Omega}{2}(-\pi e^{-\pi t}) + \frac{-\sin(\pi x)}{\pi^2} + \frac{\sin(\pi\Omega)}{\pi^2} + \frac{x}{\pi} - \frac{\Omega}{\pi} + \frac{1}{\Omega}\left[-\Omega\frac{\sin(\pi\Omega)}{\pi^2} + \left|\frac{-\cos(\pi x')}{\pi^3}\right|_0^{\Omega} - \frac{\Omega^2}{2\pi}\right]$$

$$= (x-\Omega)\pi e^{-\pi t} + \frac{\Omega}{2}(-\pi e^{-\pi t}) + \frac{-\sin(\pi x)}{\pi^2} + \frac{\sin(\pi\Omega)}{\pi^2} + \frac{x}{\pi} - \frac{\Omega}{\pi} - \frac{\sin(\pi\Omega)}{\pi^2} - \frac{\cos(\pi\Omega)}{\Omega\pi^3} + \frac{1}{\Omega\pi^3} - \frac{\Omega^2}{2\Omega\pi}$$

$$= (x-\Omega)\pi e^{-\pi t} + \frac{\Omega}{2}(-\pi e^{-\pi t}) + \frac{-\sin(\pi x)}{\pi^2} + \frac{\sin(\pi\Omega)}{\pi^2} + \frac{x}{\pi} - \frac{\Omega}{\pi} - \frac{\sin(\pi\Omega)}{\pi^2} - \frac{\cos(\pi\Omega)}{\Omega\pi^3} + \frac{1}{\Omega\pi^3} - \frac{\Omega}{2\pi}$$

$$= (x-\Omega)\pi e^{-\pi t} + \frac{\Omega}{2}(-\pi e^{-\pi t}) + \frac{-\sin(\pi x)}{\pi^2} + \frac{x}{\pi} - \frac{3\Omega}{2\pi} - \frac{\cos(\pi\Omega)}{\Omega\pi^3} + \frac{1}{\Omega\pi^3}$$

$$= (x-\Omega)\pi e^{-\pi t} + \frac{\Omega}{2}(-\pi e^{-\pi t}) + 2\pi^2 e^{-\pi t}\left(\frac{-\sin(\pi x)}{\pi^2}\right) + \frac{2\pi^2 e^{-\pi t}}{\pi}x - 2\pi^2 e^{-\pi t}\frac{3\Omega}{2\pi} - 2\pi^2 e^{-\pi t}\frac{\cos(\pi\Omega)}{\Omega\pi^3} + 2\pi^2 e^{-\pi t}.\frac{1}{\Omega\pi^3}$$

$= (x-\Omega)\pi e^{-\pi t} + \frac{\Omega}{2}(-\pi e^{-\pi t}) - 2\sin(\pi x)e^{-\pi t} + \pi(2x-3\Omega)e^{-\pi t} - \frac{1}{\Omega\pi}2\cos(\pi\Omega)e^{-\pi t} + \frac{2}{\Omega\pi}e^{-\pi t}$

$= (x-\Omega)\pi e^{-\pi t} + \frac{\Omega}{2}(-\pi e^{-\pi t}) + \frac{1}{\pi\Omega}[-2\pi\Omega\sin(\pi x) - 2\cos(\pi\Omega) + \pi^2\Omega(2x-3\Omega) + 2]e^{-\pi t}$

$= (x-\Omega)\pi e^{-\pi t} + \frac{\Omega}{2}(-\pi e^{-\pi t}) - \frac{1}{\pi\Omega}[2\pi\Omega\sin(\pi x) + 2\cos(\pi\Omega) + \pi^2\Omega(3\Omega-2x) - 2]e^{-\pi t}$

$u_0 = (x-\Omega)\pi e^{-\pi t} - \frac{\Omega}{2}\pi e^{-\pi t} + \frac{1}{\pi\Omega}[2\pi\Omega\sin(\pi x) + 2\cos(\pi\Omega) + \pi^2\Omega(3\Omega-2x) - 2]e^{-\pi t}$

$u_0 = (x-\Omega)\pi e^{-\pi t} - \frac{\Omega}{2}\pi e^{-\pi t} + \frac{1}{\pi\Omega}[2\cos(\pi\Omega) + 2\pi\Omega\sin(\pi x) + \pi^2\Omega(3\Omega-2x) - 2]e^{-\pi t}$

$u_{n+1} = \frac{1}{\Omega}\int_0^{\Omega} u_n(x,t)dx + \int_{\Omega}^{x} dx' \int_0^{x'} \frac{\partial^2 u_n(x'',t)}{\partial t^2}dx'' + \frac{1}{\Omega}\int_0^{\Omega} dx'(x' \int_1^{x'} \frac{\partial^2 u_n(x'',t)}{\partial t^2}dx'')$

$\varphi_5(x,t) = e^{-\pi t}\left[\pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \frac{(\pi x)^7}{7!} + \frac{(\pi x)^9}{9!} - \frac{(\pi x)^{11}}{19958400} + \cdots\right]$

Estimate more terms therefore

$$u(x,t) = \lim_{n\to\infty}\varphi_n(x,t) = e^{-\pi t}\sin(\pi x)$$

Which matches the accurate solution

$$u(x,t) = e^{-\pi t}\sin(\pi x) + \mu_5 t + \mu_6$$

where μ_5 *and* μ_6 are arbitrary therefore μ_5 *and* μ_6 vanish.

Table 4: Summary of results obtained in example 4

x	t	u	Exact	Error(1.0e-03)
0	0	0	0	0
0.05	0.05	0.133694525331659	0.133694525331659	0.000000000000056
0.1	0.1	0.225706844271172	0.225706844271226	0.000000000053818
0.15	0.15	0.283393778539763	0.283393778543749	0.000000003985257
0.2	0.2	0.313576432136283	0.313576432217014	0.000000080731533
0.25	0.25	0.322396941140479	0.322396941944834	0.000000804354972
0.3	0.3	0.315242477067670	0.315242482184127	0.000005116456370
0.35	0.35	0.296721574800026	0.270679760789519	0.000023880498135
0.4	0.4	0.270679671921842	0.296721598680524	0.000088867676273
0.45	0.45	0.240242625238963	0.240242903426645	0.000278187682112
0.5	0.5	0.207878816529668	0.207879576350762	0.000759821093571
0.55	0.55	0.175472204534316	0.175474063168161	0.001858633845164
0.6	0.6	0.144400277908527	0.144404428880597	0.004150972069766
0.65	0.65	0.115612286868291	0.115620875625684	0.008588757393277
0.7	0.7	0.089704366707892	0.089721018894541	0.016652186648619
0.75	0.75	0.066989212680271	0.067019739708273	0.030527028002325
0.8	0.8	0.047558829607868	0.047612129067901	0.053299460032789
0.85	0.85	0.031339573569087	0.031428732320961	0.089158751874673
0.9	0.9	0.018139243360200	0.018282839453758	0.143596093558532
0.95	0.95	0.007686384523204	0.007909971253941	0.223586730737733
1	1	0.000337742318605	0.000000000000000	0.337742318605137

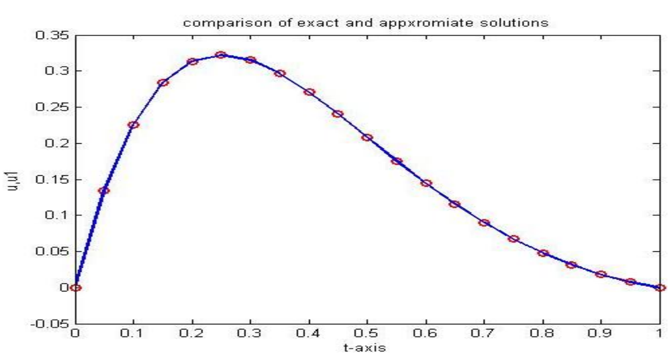

Figure 7: Graph for nonhomogeneous wave equation

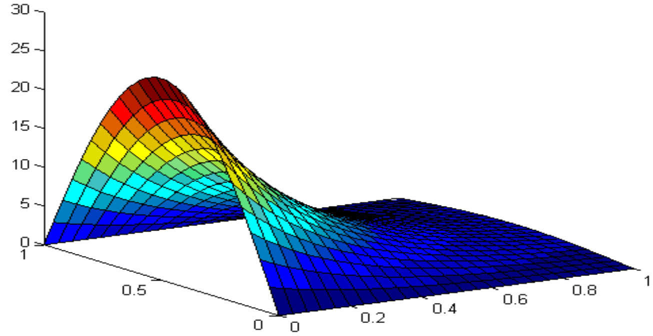

Figure 8: Graph of exact solution for nonhomogeneous wave equation in 3-D

4. CONCLUSION

Adomian decomposition method has been known to be a powerful device for solving many functional equations as algebraic equations, ordinary and partial differential equations, integral equations and so on. Here we used this method for solving nonlinear BVP. It is demonstrated that this method has the ability of solving systems of both linear and non-linear differential equations. In above problems, there was a nonlinear system and we derived the exact solutions. For non-linear systems, we usually derive a very good approximation to the solutions with the Neumann boundary conditions. It is also important that the Adomian decomposition method does not require discretization of the variables. It is not affected by computation round errors and one is not faced with necessity of large computer memory and time. Comparing the results with exact solutions, the Adomian decomposition method was clearly reliable techniques. It is important that this method unlike the most numerical techniques provides a closed form of the solution.

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